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DEGREE ONE MAPS BETWEEN GEOMETRIC 3-MANIFOLDS

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ABSTRACT. Let M and N be two compact orientable 3-manifolds, we say that $M \geq N$, if there is a degree one map from M to N. This gives a way to measure the complexity of 3-manifolds. The main purpose of this paper is to give a positive answer to the following conjecture: if there is an infinite sequence of degree one maps between Haken manifolds, then eventually all the manifolds are homeomorphic to each other. More generally, we prove a theorem which says that any infinite sequence of degree one maps between the so-called "*geometric* 3-*manifolds*" must eventually become homotopy equivalences.

1. INTRODUCTION

1.1. **The main result.** By a geometric 3-manifold we mean an orientable connected 3-manifold which is either a hyperbolic manifold, or a Seifert fibered manifold, or a Haken manifold, or a connected sum of such manifolds. We denote the class of geometric 3-manifolds by $\mathscr{G}$. The well-known geometrization conjecture of W. Thurston states that $\mathscr{G}$ represents all compact orientable connected 3-manifolds [18].

Denote the class of closed manifolds in $\mathscr{G}$ by $\mathscr{G}_c$. Let $\sim$ be the equivalence relation on $\mathscr{G}_c$ defined by $M \sim N$ iff M is homotopically equivalent to N. Let $\mathscr{G}_c/\sim$ denote the set of equivalence classes in $\mathscr{G}_c$. We define a relation $\geq$ on $\mathscr{G}_c/\sim$ by $[M] \geq [N]$ iff there is map $f\colon M \to N$ with $\deg f = 1$ for some orientations on M and N. Since a homotopy equivalence is a degree one map, $\geq$ is a well-defined relation on $\mathscr{G}_c/\sim$. It can be proved (Theorem 2.1) that $\geq$ is a partial order on $\mathscr{G}_c/\sim$.

Our main theorem here shows that any infinite decreasing sequence with respect to this partial order must eventually stabilize:

Theorem 3.9. *Let* $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots$ *be an infinite sequence of maps between compact oriented manifolds such that for all* i,

(1) $M_i \in \mathscr{G}_c$,

(2) $\deg f_i = 1$.

Then for i *sufficiently large,* M_i *is homotopically equivalent to* M^{i+1}, *and* f_i *is a homotopy equivalence.*

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Similar result holds for manifolds with boundary, i.e. manifolds in $\mathcal{G}$. See Theorem 3.15.

We denote by $\mathcal{G}_1$ the class of manifolds in $\mathcal{G}$ which do not have a lens space as a connected summand. It can be shown that in $\mathcal{G}_1$ homotopy equivalent $(\mathrm{rel}\,\partial)$ manifolds are homeomorphic. Thus $\geq$ defines a partial order on $\mathcal{G}_1$, and any infinite decreasing sequence in $\mathcal{G}_1$ will eventually consist of homeomorphic manifolds. In particular, since all Haken manifolds are contained in $\mathcal{G}_1$, we have the following corollary:

Corollary 3.16. *Let* $(M_1, \partial M_1) \xrightarrow{f_1} (M_2, \partial M_2) \xrightarrow{f_2} \cdots$ *be an infinite sequence of maps between compact oriented manifolds such that for all* i,

(1) M_i *is Haken,*

(2) $\deg f = 1$.

Then for i *sufficiently large,* M_i *is homeomorphic to* M_{i+1} *and* f_i *is homotopic to a homeomorphism.*

This paper is organized as follows:

In §2, we prove our result for closed Haken manifolds. Since Haken manifolds are decomposed into Seifert fibered pieces and hyperbolic pieces, we first deal with such pieces (Theorem 2.2 and Theorem 2.3). The hyperbolic manifold case can be treated easily using Gromov's norm. The Seifert fibered manifold case is proved using the Euler number of a Seifert fibered space. For general closed Haken manifold, we use Jaco-Shalen and Johannson's theory on maps from Seifert fibered spaces into Haken manifolds [5 and 6]. They proved that any "*nondegenerate map*" can be deformed into the so-called "*Characteristic submanifold*." We will mainly use definitions of [5]. The definition of nondegenerate maps will be given in the proof of Lemma 2.13. Using this theory we can show that under certain conditions, a nondegenerate map between two Haken manifolds can be deformed into a "nice" map. For degenerate maps, we define a complexity for a Haken manifold by using Gromov's norm and the number of Seifert fibered pieces in the torus decomposition. We prove that under certain condition this complexity must strictly decrease once the map is degenerate. Therefore, the infinite sequence of maps must eventually become nondegenerate. Gromov's norm was used in a similar way by T. Soma in [15], where he proved that every sequence of preimage knots is finite.

We consider reducible manifolds in §3. Since π_2 is no longer zero, some cut and paste techniques do not work as before. We get around this problem by allowing maps which are "*almost defined*" on a manifold, namely, they are defined except at a few "holes" (pairs of open 3-balls) and satisfy certain conditions on the boundary of these holes. We then define a notion of degree for this kind of map. This allows us to reduce this later case to the previous one. At the end of §3, we consider manifolds with boundary by using their doubles.

This paper is the main part of my thesis, and as so many ideas belong to my advisor Professor Cameron Gordon. I would also like to thank the referee for his many valuable suggestions.

1.2. **Notations and preliminaries.** For topological spaces X and Y, $X \cong Y$ means that X is homeomorphic to Y, $X \simeq Y$ means that X is homotopy equivalent to Y. Similar notations are used for pairs of spaces.

For two maps f and g, $f \simeq g$ means that f and g are homotopic maps. If

f and g are maps of pairs, $f \simeq g$ as maps of pairs means they are homotopic as maps of pairs. If f_t $(0 \leq t \leq 1)$ is a homotopy on X which is fixed outside a subset A of X, we say that f_t is a homotopy supported on A.

The volume of a Riemannian manifold H is denoted by $v(H)$. Throughout, T always denotes a torus, K a Klein bottle and I a closed interval.

For a proper map between two connected oriented manifolds M and N of the same dimension, the definition of the degree of f is standard. If M is not connected, then we define $\deg f = \Sigma \deg f|M_i$, where $\{M_i\}$ are the connected components of M.

The following lemma is an easy corollary of the definition of degree:

Lemma 1.1. *If $f : M \to N$ is a map between two n-manifolds such that f is transverse to a p.l. $(n-1)$-submanifold F of N, then $f^{-1}(F)$ is a p.l. $(n-1)$-submanifold of M, and $\deg\{f^{-1}(F) \xrightarrow{f|} F\} = \deg f$ for some suitable orientation of $f^{-1}(F)$.*

The following lemma is standard for degree one maps. The proof of (1) uses a standard covering space argument (see [2, 15.12]). The proof of (2) uses the Poincaré duality and the naturality of the cap product [10].

Lemma 1.2. *Let $f : (M, \partial M) \to (N, \partial N)$ be a degree one map of compact oriented manifolds (with possibly empty boundary), then:*

(1) $f_* : \pi_1(M) \to \pi_1(N)$ is an epimorphism.

(2) $f_* : H_k(M) \to H_k(N)$ is an epimorphism, $f^* : H^k(N) \to H^k(M)$ is a monomorphism. And the same is true if we use homology rel-∂.

Similar results hold for a map of degree d. In particular, the following lemma is very useful in this paper. Note that it implies an analogue of Theorem 3.9 for surfaces.

Lemma 1.3. *Let*

$$f : (M, \partial M) \to (N, \partial N)$$

be a map of degree d, then $[\pi_1(N) : f_*(\pi_1(M))]$ divides d.

We recall some standard results on Haken manifolds. For a closed Haken manifold M, the torus decomposition theorem of Jaco-Shalen and Johannson together with the uniformization theorem of Thurston say that there is a collection of incompressible tori $W \subset M$, unique up to ambient isotopy, which cuts M into Seifert fibered manifolds and hyperbolic manifolds of finite volume. Denote the regular neighborhood of W by $W \times [-1, 1]$ with $W \times \{0\} = W$. We write $M - W \times (-1, 1) = H_M \cup S_M$, where H_M is the union of the finite volume hyperbolic manifold components, and S_M is the union of the Seifert fibered manifold components. Therefore M has the picture shown in Figure 1, where S indicates a Seifert fibered manifold, and H indicates a hyperbolic manifold of finite volume. We also define Σ_M to be the union of S_M and all components $T_j \times [-1, 1]$ of $W_M \times [-1, 1]$ such that $T_j \times \{\pm 1\} \subseteq \partial H_M$. It is the shaded part in Figure 1.

The following is a special case of the characteristic pair theorem [5].

Theorem 1.4 (Jaco-Shalen). *If f is a nondegenerate map of a Seifert pair $(S, \varnothing)$ into a Haken manifold pair $(M, \varnothing)$, then there exists a map f_1 of S into M, homotopic to f, such that $f_1(S) \subseteq \Sigma_M$.*

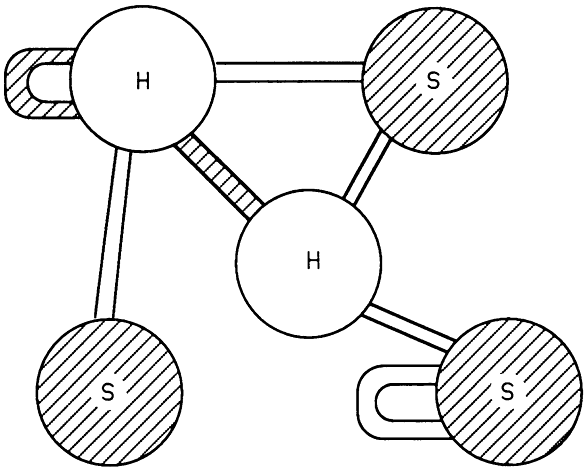

FIGURE 1

We will also use the concepts of residual finiteness and Hopficity of a group (see [2, Chapter 15] for definitions). It is true that finitely generated residually finite groups are Hopfian [2, 15.17]. It is also proved that the fundamental groups of geometric 3-manifolds are residually finite and thus Hopfian [18, 3.3 and 3]. Surface groups, Fuchsian groups are all residually finite and thus Hopfian [2, Chapter 5].

Next we talk about Gromov's norm. Let M be a compact, oriented n-manifold with boundary (possibly empty). The *Gromov's norm* of M is denoted by $\|M\|$. See [19] for the definition.

The following theorem is immediate from the definition:

Theorem 1.5. *Let $f: (M, \partial M) \to (N, \partial N)$ be a map of degree d, then $\|M\| \geq d\|N\|$. Furthermore, if f is a covering map, then $\|M\| = d\|N\|$.*

Corollary 1.6. *If M has a self-map of degree $d \geq 2$, then $\|M\| = 0$.*

The following theorem is due to Gromov:

Theorem 1.7 (Gromov). *Let M^n be any closed oriented hyperbolic manifold of dimension $n > 1$, then $\|M\| = v(M)/v_n$ where v_n is the supremum of the volumes of straight n-simplices in H_n.*

In [19, Chapter 6], Thurston generalized Gromov's theorem to relative versions and to strict versions. The following is one form of Thurston's generalization:

Theorem 1.8. *Let M, N be compact, oriented 3-manifolds whose interiors admit hyperbolic structures of finite volume. If $f : (M, \partial M) \to (N, \partial N)$ is a map of degree $d > 0$, then $v(M) \geq dv(N)$. If the equality holds, then f is homotopic to a local isometry, and therefore a covering map. When $d = 1$, in particular, this implies that f is homotopic to a homeomorphism.*

For 3-manifolds, Gromov's norm is additive under connected sum [1, p. 10]. It is also additive when splitting along incompressible tori and is subadditive when splitting along compressible tori [14]. Using these and the relative version of Gromov's theorem, we have $\|M\| = v(H_M)/v_3$ for any Haken manifold M.

2. Case for closed Haken manifolds

2.1. **The partial order "$\geq$".** We prove that $\geq$ we have defined is a partial order.

Theorem 2.1. *The relation $\geq$ is a partial order on $\mathscr{G}_c/\sim$.*

Proof. Let M_1 and M_2 be manifolds in $\mathscr{G}_c$ such that $[M_1] \geq [M_2]$ and $[M_2] \geq [M_1]$. Let $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_1$ be two degree one maps. Then $f_1 \circ f_2$ is a degree one map from M_1 to itself. By Lemma 1.2, the map induces an epimorphism on π_1. By the Hopficity of $\pi_1(M_1)$, the epimorphism must be an isomorphism. Since it is a degree one map, it must be a homotopy equivalence [16, 3.6]. It follows that $f_1 \circ f_2$ induces isomorphisms on all the homotopy groups π_i. Thus f_1 induces epimorphisms on all the π_i's. Similarly, we can show that f_1 induces monomorphisms on all the homotopy groups by considering $f_2 \circ f_1$. Therefore, f_1 induces isomorphisms on all the homotopy groups and thus is a homotopy equivalence. So we have proved that $[M_1] = [M_2]$.

Remark. The above theorem shows that to prove Theorem 3.9, it is enough to prove that $M_{k_i} \sim M_{k_{i+1}}$ for a subsequence M_{k_i}. We will use this argument in our proof often.

2.2. **Case for hyperbolic manifolds and Seifert fibered spaces.**

Theorem 2.2. *Let $(H_1, \partial H_1) \xrightarrow{f_1} (H_2, \partial H_2) \xrightarrow{f_2} \cdots$ be an infinite sequence of maps between compact oriented 3-manifolds such that for each i,*

1. $\operatorname{Int} H_i$ *is a complete hyperbolic manifold with finite volume.*
2. $\deg f_i = 1$.

Then for i large enough, $H_i \cong H_{i+1} \cong \cdots$, and f_i is homotopic to a homeomorphism.

Proof. Since $\|H_i\| = v(H_i)/v_3$, we have

$$v(H_1) \geq v(H_2) \geq \cdots.$$

By [19, 6.6.3], for i large enough, $v(H_i) = v(H_{i+1}) = \cdots$. By Theorem 1.8, $H_i \cong H_{i+1} \cong \cdots$, and f_i is homotopic to a homeomorphism.

Theorem 2.3. *Let $(S_1, \partial S_1) \xrightarrow{f_1} (S_2, \partial S_2) \xrightarrow{f_2} \cdots$ be an infinite sequence of maps between compact oriented 3-manifolds such that for each i,*

1. S_i *is a Seifert fibered manifold with infinite π_1.*
2. $\deg f_i = 1$.

Then for i large enough, $S_i \cong S_{i+1} \cong \cdots$, and f_i is homotopic to a homeomorphism.

Proof. Fix a Seifert fibration of S_1 with regular fiber h_1.Then $\langle f_{1*}(h_1)\rangle$ is a cyclic subgroup of $\pi_1(S_2)$. Since $\pi_1(S_2)$ is torsion free [2, 9.9], $\langle f_{1*}(h_1)\rangle$ is isomorphic to either Z or $\{0\}$.

If $\langle f_{1*}(h_1)\rangle \cong Z$, then it is an infinite cyclic normal subgroup of $\pi_1(S_2)$. (It is normal because f_{1*} is onto.) By [4, VI.11.e], there is a Seifert fibration of S_2 with regular fiber h_2 such that $f_{1*}(h_1) \in \langle h_2\rangle$. If $f_{1*}(h_1) = \{0\}$, taking any fibration of S_2 and we still have $f_{1*}(h_1) \in \langle h_2\rangle$. Similarly, there exists a fibration of S_3 with regular fiber h_3 such that $f_{2*}(h_2) \in \langle h_3 \rangle$, and so on. So we get an induced infinite sequence of epimorphisms:

$$\pi_1(S_1)/\langle h_1 \rangle \xrightarrow{\hat{f}_{1*}} \pi_1(S_2)/\langle h_2 \rangle \xrightarrow{\hat{f}_{2*}} \cdots .$$

We denote O_i as the base orbifold of the Seifert fibered manifold S_i. The above infinite sequence becomes the infinite sequence of epimorphisms between π_1 of 2-dimensional orbifolds

$$\pi_1(O_1) \xrightarrow{\hat{f}_{1*}} \pi_1(O_2) \xrightarrow{\hat{f}_{2*}} \cdots .$$

Thus by Lemma 2.5 below,

$$-\chi(O_1) \geq -\chi(O_2) \geq \cdots$$

by Lemma 2.6(a), $-\chi(O_1) = -\chi(O_2) = \cdots$ for i sufficiently large. By Lemma 2.6(b), we can pass to a subsequence such that $O_1 \cong O_2 \cong \cdots$. By the Hopficity of Fuchsian groups, the induced epimorphisms $\hat{f}_{i_*}$ are isomorphisms.

The commutative diagram

$$\begin{array}{ccc} \pi_1(S_i) & \xrightarrow{f_{i_*}} & \pi_1(S_{i+1}) \\ {\scriptstyle p_{i_*}} \downarrow & & {\scriptstyle p_{i+1_*}} \downarrow \\ \pi_1(S_i)/\langle h_i \rangle & \xrightarrow{\hat{f}_{i_*}} & \pi_1(S_{i+1})/\langle h_{i+1} \rangle \end{array}$$

tells us that each epimorphism f_{i_*} is an isomorphism. This can be proved as in the following: If $f_{i_*}(\alpha) = 1$ for some $\alpha \neq 1$, then $p_{i_*}(\alpha) = 1$, and thus $\alpha \in \langle h_i \rangle$. Let $\alpha = h_i^s$, $s \neq 0$, then $(f_{i_*}(h_i))^s = f_{i_*}(\alpha) = 1$. Since $\pi_1(S_{i+1})$ is torsion free, $f_{i_*}(h_i) = 1$. But if f_{i_*} is onto, so there exists $\beta \in \pi_1(S_i)$, $f_{i_*}(\beta) = h_{i+1}$. This implies that $p_{i_*}(\beta) = 1$, so $\beta \in \langle h_i \rangle$. Therefore $f_{i_*}(\beta) = 1$, a contradiction.

So we have proved that $\pi_1(S_i) \cong \pi_1(S_{i+1})$, and therefore $S_i \cong S_{i+1}$ [11]. Since each S_i is aspherical, f_i is a homotopy equivalence, thus is a simple homotopy equivalence since the Whitehead group of $\pi_1(S_i)$ vanishes [12]. By a result of Turaev [20, Theorem 1.6], f_i is homotopic to a homeomorphism.

I would like to thank K. Miyazaki for informing me of the result of Turaev.

If we allow Seifert fibered spaces of finite π_1 in the above theorem, then eventually all the $\pi_1(M_i)$'s have the same order, and therefore isomorphic to each other by Lemma 1.2. If we rule out the lens space, then $M_i \cong M_{i+1}$ [11]. So we have

Theorem 2.4. *In Theorem 2.3, if we allow manifolds with finite π_1 which are not homeomorphic to a lens space, then for i large enough, $M_i \cong M_{i+1}$.*

Lemma 2.5. *Let O_1, O_2 be 2-dimensional compact orbifolds with*

(1) *An epimorphism $\alpha : \pi_1(O_1) \to \pi_2(O_2)$.*

(2) *∂O_1 and ∂O_2 are both empty or both nonempty.*

(3) *O_2 is a good orbifold.*

Then $-\chi(O_1) \geq -\chi(O_2)$.

Proof. Since O_2 is a compact good 2-dimensional orbifold, there is a finite orbifold covering $p_2 : F_2 \to O_2$ such that F_2 is a surface. We may assume that

F_2 is orientable by taking a double covering if necessary. Let $p_1 : \widetilde{O}_1 \to O_1$ be the orbifold covering of O_1 corresponding to the subgroup $\alpha^{-1}(\pi_1(F_2))$ of $\pi_1(O_1)$ and let F_1 be the base surface of $\widetilde{O}_1$, we have the commutative diagram:

$$\begin{array}{ccc} \pi_1(\widetilde{O}_1) & \xrightarrow{\tilde{\alpha}} & \pi_1(F_2) \\ {\scriptstyle p_{1*}}\downarrow & & {\scriptstyle p_{2*}}\downarrow \\ \pi_1(O_1) & \xrightarrow{\alpha} & \pi_1(O_2) \end{array}$$

Note that p_1 is a finite cover, for it has the same covering degree as p_2. Hence O_1 and F_1 are compact. Since $\tilde{\alpha}$ is onto, it induces an epimorphism on H_1, i.e. the abelianization of π_1. But $H_1(F_2)$ is torsion free, and $H_1(F_1)$ is the free part of $H_1(O_1)$. So

$$\beta_1(F_1) \geq \beta_1(F_2), \quad \text{where } \beta_1 \text{ is the first Betti number}.$$

F_1, being the underlying surface of O_1, may have boundary even when O_1 does not have one. So we consider the following two cases:

Case 1. If ∂O_1 and ∂O_2 are both nonempty, then ∂F_1 and ∂F_2 are both nonempty. Hence

$$-\chi(F_1) = -1 + \beta_1(F_1), \qquad -\chi(F_2) = -1 + \beta_1(F_2).$$

So $-\chi(F_1) \geq -\chi(F_2)$.

Case 2. If $\partial O_1 = \partial O_2 = \varnothing$, then $\partial F_2 = \varnothing$,

$$-\chi(F_1) = \begin{cases} -2 + \beta_1(F_1) & \text{if } F_1 \text{ is closed and orientable}, \\ -1 + \beta_1(F_1) & \text{otherwise}, \end{cases}$$
$$-\chi(F_2) = -2 + \beta_1(F_2).$$

So $-\chi(F_1) \geq -\chi(F_2)$.

Since $-\chi(\widetilde{O}_1) = -\chi(F_1) + \Sigma(1 - 1/q_i) + \frac{1}{2}\Sigma(1 - 1/r_j)$ [13, p. 427], so

$$-\chi(\widetilde{O}_1) \geq -\chi(F_1) \geq -\chi(F_2).$$

So we conclude that $-\chi(O_1) = \frac{1}{d}(-\chi(\widetilde{O}_1)) \geq \frac{1}{d}(-\chi(F_2)) = -\chi(O_2)$.

The following lemma is easily proved using the formula for $\chi(O)$ given in [13], the proof is omitted.

Lemma 2.6. (a) *The set* $S = \{-\chi(O) : O$ *is a compact, connected* 2*-dimensional orbifold*$\}$ *is a well-ordered subset of reals.*

(b) *For a fixed rational number* r, *there are at most finitely many* 2*-dimensional orbifolds* O *with* $-\chi(O) = r$.

2.3. **Closed Haken manifold case.** First we consider maps from $T \times I$ to a 3-manifold M which induces an injective map on π_1. In most cases the image of the map can be pushed into ∂M. Special care must be taken for the twisted I-bundle over the Klein bottle.

Let $K \tilde{\times} I$ denote the twisted I-bundle over the Klein bottle. It is doubly covered by $T_1 \times I$, where T_1 is a torus. Since this covering space has the fundamental group carried by $\partial(K \tilde{\times} I)$, we have

Lemma 2.7. *Any map* f *from* $(T \times I, T \times \partial I)$ *to* $(K \tilde{\times} I, \partial(K \tilde{\times} I))$ *lifts to the double covering* $T_1 \times I$.

This lemma together with [2, 13.6] implies

Lemma 2.8. *Suppose that* M *is a compact orientable* 3*-manifold which is* P^2*-irreducible and sufficiently large. Let* $f: (T \times I, T \times \partial I) \to (M, \partial M)$ *be a map which induces an injective map on* π_1. *Then there is a homotopy* $f_t: (T \times I, T \times \partial I) \to (M, \partial M)$ *such that* $f_0 = f$ *and either*

(1) $f_1(T \times I) \subseteq \partial M$, *or*
(2) $M \cong T \times I$, *and* f_1 *is a covering, or*
(3) $M \cong K \tilde{\times} I$, *and* f_1 *is a covering of even degree.*

Corollary 2.9. *If* $M \ncong T \times I$ *in the above lemma, then either* $\deg f = 0$ *or* $\deg f$ *is even and* $M \cong K \tilde{\times} I$.

Before we prove Lemma 2.11, we first quote a lemma from [4, IX.1].

Lemma 2.10. *Let* p *be an orientable closed surface. Then any closed incompressible surface in* $P \times I$ *is isotopic to* $P \times \{r\}$ *for some* r.

Lemma 2.11. *Suppose that* M, N *are closed Haken manifolds with* $H_M \cong H_N$. *Let* $f: M \to N$ *be a map such that* $\deg f \neq 0$ *and* $f(\Sigma_M) \subset \operatorname{Int} \Sigma_N$. *Then there exists a homotopy* f_t *of* f $(0 \le t \le 1)$, *such that* $f_0 = f$, $f_t|_{\Sigma_M} = f|_{\Sigma_M}$, *and* $f_1: (H_M, \partial H_M) \to (H_N, \partial H_N)$ *is a homeomorphism.*

Proof. If $H_M = \varnothing$, there is nothing to prove. So we assume that $H_M \neq \varnothing$. In this case, it is easy to see that $\deg f = 1$ using $\|M\| = \|N\| \neq 0$.

Take any component T of ∂H_N. We shall show that each component of $f^{-1}(T)$ is parallel to some component of W_M after homotoping f fixing $f|_{\Sigma_M}$.

Since $f(\Sigma_M) \subset \operatorname{Int} \Sigma_N$, $f^{-1}(T) \cap \Sigma_M = \varnothing$. Since $\partial \Sigma_M$ is incompressible, using a standard cut and paste argument, we may assume that after changing f by a homotopy fixing $f|_{\Sigma_M}$, f is transverse to T, and $f^{-1}(T)$ is a collection of incompressible surfaces in $M - \Sigma_M$. We claim that each component of $f^{-1}(T)$ is a torus.

Suppose that P_1 is a component of $f^{-1}(T)$ with $\text{genus}(P_1) > 1$. Since $P_1 \subset M - \Sigma_M$, which is a disjoint union of H_M and some $T \times I$ in $W_M \times I$, $P_1 \subset \operatorname{Int} H_M$. Let $\phi: H_M \to H_N$ be a homeomorphism, and $P_1' = \phi(P_1)$. By the same reason as before, we may assume that f is transverse to P_1', and $f^{-1}(P_1')$ is a collection of incompressible surfaces in $M - \Sigma_M$. Since

$$\deg\{f^{-1}(P_1') \xrightarrow{f} P_1'\} = \deg f = 1$$

under suitable orientation of $f^{-1}(P_1')$ and P_1', we can take a connected component P_2 of $f^{-1}(P_1')$ such that $\deg\{P_2 \xrightarrow{f|_{P_2}} P_1'\} \neq 0$. Since $\|P_2\| \geq \|P_1'\| > 0$, $\text{genus}(P_2) > 1$. Hence $P_2 \subset \operatorname{Int} H_M$. We repeat this process to get an infinite sequence of surfaces $P_1, P_2, \ldots$ such that:

1. $\text{genus}(P_i) > 1$.
2. $P_i \subset \operatorname{Int} H_M$ is incompressible.
3. $(H_M, P_i) \cong (H_N, P_i')$ under the homeomorphism ϕ.
4. f is transverse to P_i' and $\deg\{P_{i+1} \xrightarrow{f|_{P_{i+1}}} P_i'\} \neq 0$.

Denote $P'_0 = T$, $P_0 = \phi^{-1}(P'_0) \subset \partial H_M$. Using $f(P_i \cap P_j) \subset P'_{i-1} \cap P'_{j-1} = \phi(P_{i-1} \cap P_{j-1})$ and $P_0 \cap P_k = \varnothing$ for all $k > 0$, we can prove inductively that $P_i \cap P_j = \varnothing$ for all $j > i \geq 0$. Since H_M is compact, there must be some $i \neq j$ such that $P_i \| P_j$ in H_M [4, III.20]. Let i be the minimum i such that $P_i \| P_j$ in H_M for some $j > i$. We want to show that $P_{i-1} \| P_{j-1}$ in H_M to get a contradiction.

Let $k_l = \deg\{P_l \xrightarrow{\phi^{-1} \circ f} P_{l-1}\}$. Let k be the degree of the composition map

$$P_j \xrightarrow{\phi^{-1} \circ f} P_{j-1} \xrightarrow{\phi^{-1} \circ f} \cdots \xrightarrow{\phi^{-1} \circ f} P_i .$$

Then $k_l \neq 0$, and $k = k_{i+1} \cdots k_j$. Using Gromov's norm, we have

$$\|P_j\| \geq k_j \|P_{j-1}\| \geq k_j k_{j-1} \|P_{j-2}\| \geq \cdots \geq k \|P_i\| .$$

Since $P_j \cong P_i$, $\|P_j\| = \|P_{j-1}\| = \cdots = \|P_i\|$, and $k = k_{i+1} = \cdots = k_j = 1$. Therefore $P_j \cong P_{j-1} \cong \cdots \cong P_i$. By the Hopficity of surface groups, all these degree one maps between the surfaces induce isomorphisms on π_1.

Let $P \times [0, 1]$ be the region bounded by $P_i \cup P_j$ in H_M, with $P \times \{0\} = P_i$. Any component of $f^{-1}(P'_{i-1} \cup P'_{j-1})$ in $P \times (0, 1)$ (if any) is isotopic to $P \times \{r\}$ for some r by Lemma 2.10. By taking the "rightmost" component of $f^{-1}(P'_{i-1})$ in $P \times [0, 1]$, we get an "innermost subinterval" $P \times [a, b] \subset P \times [0, 1]$, such that

$$f(P \times \{a\}) = P'_{i-1}, \quad f(P \times \{b\}) = P'_{j-1},$$

and

$$f(P \times (a, b)) \cap (P'_{i-1} \cup P'_{j-1}) = \varnothing .$$

Let N_1 be the closure of the component of N cut along $P'_{i-1} \cup P'_{j-1}$ in which $f(P \times [a, b])$ lies. We have a map of pairs

$$(P \times [a, b], P \times \{a, b\}) \xrightarrow{f|P \times [a,b]} (N_1, \partial N_1) .$$

In the commutative diagram

$$\begin{array}{ccc} \pi_1(P \times \{b\}) & \xrightarrow{(f|)_*} & \pi_1(P'_{j-1}) \\ \downarrow{i_*} & & \downarrow{i_*} \\ \pi_1(P \times [a, b]) & \xrightarrow{(f|)_*} & \pi_1(N_1) \end{array}$$

$(f|_{P \times b})_*$ is injective. The incompressibility of P'_{j-1} implies that $i_* : \pi_1(P'_{j-1}) \to \pi_1(N_1)$ is also injective. Clearly $i_* : \pi_1(P \times b) \to \pi_1(P \times [a, b])$ is an isomorphism. These imply that $(f|P \times [a, b])_*$ is injective. On the other hand, $\deg\{f|P \times [a, b]\} = \deg\{P \times \{b\} \xrightarrow{f} P'_{j-1}\} = 1$. So $(f|P \times [a, b])_*$ is surjective. By Waldhausen's theorem, $N_1 \cong P \times [a, b]$.

By Lemma 2.10, none of the tori in ∂H_N can be contained in N_1. Therefore the submanifold N_1 must be contained in H_N. Hence $\phi^{-1}(N_1) \cong N_1 \cong P \times [a, b]$, giving $P_{i-1} \| P_{j-1}$ in H_M. This contradicts the minimality of i.

So we have proved that $f^{-1}(T)$ must be a union of incompressible tori in $H_M \cup \bigcup_j T_j \times I$. Since each torus in H_M is ∂-parallel, we can change f by a homotopy (which is supported on a regular neighborhood of H_M) to push these tori to ∂H_M. We do this for each T in ∂H_N. Thus $f^{-1}(\partial H_N)$ consists of a union of parallel copies of W_M.

Each component E_i of $f^{-1}(H_N)$ is some component of M cut along $f^{-1}(\partial H_N)$. Since $f(\Sigma_M) \cap H_N = \varnothing$, $f^{-1}(H_N) \subset M - \Sigma_M = H_M \cup \bigcup T_j \times I$. It follows that each E_i is some component(s) of H_M attached by some $T_j \times I$ or is just some $T_j \times I$.

For each component H of H_N, $\deg\{f^{-1}(H) \xrightarrow{f|} H\} = \deg f = 1$. Since any map $T \times I \to H$ has degree zero using Gromov's norm, $f^{-1}(H)$ must contain some component(s) of H_M. Since $H_M \cong H_N$, they have the same number of components. Therefore $f^{-1}(H)$ contains exactly one component H' of H_M, and this component maps into H_N with degree one. By Theorem 1.8, after a homotopy supported on a regular neighborhood of H', f maps H' onto H homeomorphically. We do this for each component H of H_N to get the conclusion of the lemma.

Next we prove Lemma 2.13, which plays a key role dealing with degenerate maps. We first define the characteristic q^2-fold cover for a torus T to be the cover corresponding to the subgroup $q(Z \oplus Z)$ of $\pi_1(T) \cong Z \oplus Z$. It does not depend on the choice of the base of $\pi_1(T)$. We also use Thurston's notion of the generalized Dehn surgery coefficients. The precise definition is in [19].

We first prove Lemma 2.12, which generalizes a theorem of Thurston [19, 6.5.6].

Lemma 2.12. *Let H be a compact 3-manifold with boundary whose interior has a complete hyperbolic structure of finite volume. Let T be a component of ∂H and λ be an essential simple closed curve on T. Attach a solid torus V to T such that the meridian of V is identified with λ, and denote the resulting manifold as $\widehat{H} = \widehat{H_\lambda}$. Then $\|\widehat{H}\| < \|H\|$.*

Proof. By [8 or 3, 4.1], for all but finitely many primes q, there is a finite, connected, regular cover $p_q : \widetilde{H}_q \to H$, such that for each component $\widetilde{T}$ of $p_q^{-1}(T)$, $p_q|_{\widetilde{T}} : \widetilde{T} \to T$ is the characteristic q^2-fold cover. In particular, $p_q^{-1}(\lambda)$ has q parallel copies of components in $\widetilde{T}$, and each such component $\tilde{\lambda}$ covers λ q times. Attach a solid torus V to each $\widetilde{T}$ in $p_q^{-1}(T)$ such that the meridian of V is identified with $\tilde{\lambda}$. Denote the resulting manifold by $\widehat{\widetilde{H}}_q$. Then p_q extends to a branched cover $\hat{p}_q : \widehat{\widetilde{H}}_q \to \widehat{H}$ branched over the core of the attached solid tori V's, each branching index is q.

Extend λ to a base λ, μ of $\pi_1(T)$. By the hyperbolic Dehn surgery theorem [19, 5.8], when q is large enough, $H_{(q,0),\infty,\dots,\infty}$ has a hyperbolic structure. (Here $(q, 0)$ is the surgery coefficient for T with the base λ, μ, and ∞'s are for other boundary components.) That is to say, $H_{(q,0),\infty,\dots,\infty}$ is topologically the manifold $H_{(1,0),\infty,\dots,\infty} = \widehat{H}$, and the completed hyperbolic structure has singularities at the core of V with cone angle $2\pi/q$. Therefore it induces a non-singular hyperbolic structure on $\widehat{\widetilde{H}}_q$. $\widetilde{H}_q$, being a finite cover of the hyperbolic manifold H_q, is hyperbolic. By [19, 6.5.6], $v(\widehat{\widetilde{H}}_q) < v(\widetilde{H}_q)$, so $\|\widehat{\widetilde{H}}_q\| < \|\widetilde{H}_q\|$. Let m be the covering degree of p_q, then $\|\widetilde{H}_q\| = m\|H_q\|$. The branched cover $\hat{p}_q$ also has degree m. Thus $\|\widehat{\widetilde{H}}_q\| \geq m\|\widehat{H}\|$.

Hence we conclude that

$$\|\widehat{H}\| \leq \frac{1}{m}\|\widehat{\widetilde{H}}_q\| < \frac{1}{m}\|\widetilde{H}_q\| = \|H\|.$$

Before we prove Lemma 2.13, let us recall that $s(M)$ is the number of Seifert fibered pieces in the torus decomposition of M. For two components R_1 and R_2 of Σ_M or H_M, we say that R_1 is *adjacent* to R_2 if they are connected by some $T \times I$ in $W_M \times [-1, 1]$. We will also use the following definition of degenerate maps given in [5].

Definition. Let (S, F) be a connected Seifert pair, and let (N, T) be a connected 3-manifold pair. A map $f: (S, F) \to (N, T)$ is said to be *degenerate* if either

(1) the map f is inessential as a map of pairs, or

(2) the group $\mathrm{Im}(f_* : \pi_1(S) \to \pi_1(N)) = \{1\}$, or

(3) the group $\mathrm{Im}(f_* : \pi_1(S) \to \pi_1(N))$ is cyclic and $F = \varnothing$, or

(4) the map $f|\gamma$ is homotopic in N to a constant map for some fiber γ of (S, F).

For our map between M and N, we say that $f|\Sigma_M$ is degenerate if the map of pairs $f : (S, \varnothing) \to (N, \varnothing)$ is degenerate for a connected component S of Σ_M. Otherwise we say $f|\Sigma_M$ is *nondegenerate.*

Lemma 2.13. *Suppose that* M, N *are closed, oriented Haken manifolds with* $H_M \cong H_N$. *Let* $f: M \to N$ *be a map such that* $\deg f$ *is odd. Then* $s(M) \geq s(N)$. *Furthermore, if* $f|\Sigma_M$ *is degenerate, then* $s(M) > s(N)$.

Proof. We consider two cases.

Case 1. $f|\Sigma_M$ is nondegenerate.

By Theorem 1.4, $f|\Sigma_M$ can be homotoped into $\mathrm{Int}\,\Sigma_M$. Extend this homotopy to be a homotopy of f on M supported on a regular neighborhood of Σ_M. So we have $f(\Sigma_M) \subset \mathrm{Int}\,\Sigma_N$. By Lemma 2.11, we can change f by a homotopy such that $f(H_M) \subset H_N$.

Let S be a component of S_N. Using a standard cut and paste argument and the fact that $\partial\Sigma_M$ and ∂H_M are incompressible, we can change f by a homotopy fixing $f|_{H_M \cup \Sigma_M}$ such that $f^{-1}(\partial S)$ is a collection of incompressible surfaces in M. Since $f^{-1}(\partial S)$ lies in $M - (H_M \cup \Sigma_M)$, it must be a union of parallel copies of some tori in W_M. Now $f^{-1}(S)$ consists of some components of M cut along $f^{-1}(\partial S)$. Each such component is either some components of S_M attached along some components of $W_M \times I$ or homeomorphic to $T \times I$. If $S \ncong T \times I$, by Corollary 2.9, any map from $T \times I$ to S has even degree. Therefore $f^{-1}(S)$ must contain some component of S_M. Thus $s(M) \geq s(N)$. If $S \cong T \times I$, N must be a fibered manifold with fiber a torus and a hyperbolic glueing map. So $s(N) = 1$. Since $H_M \cong H_N \cong \varnothing$, $s(M) \geq 1$. Hence in any case we have proved that $S(M) \geq s(N)$.

Case 2. $f|\Sigma_M$ is degenerate. We prove this case by inducting on $s(M)$.

If $s(M) = 0$, using the same argument as in Case 2B of the following, we can prove that $\|M\| > \|N\|$. This contradicts the assumption $H_M \cong H_N$, so this case never happens.

Now we assume that $s(M) > 0$, and the lemma is true for all integers $<$ $s(M)$.

We consider two subcases:

Case 2A. For any component S of Σ_M adjacent to H_M, $f|_S$ is nondegenerate.

We want to construct a manifold M_1 with $s(M_1) < s(M)$ and a map from M_1 to N so that we can use induction. First, we construct a space X, and maps α, β such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & X & \end{array}$$

commutes after changing f by a homotopy.

Let B be the union of all S in Σ_M such that $f|_S$ is degenerate. Let $G = \overline{M - B}$. Then $M = B \cup G$, $B \cap G = \bigcup_j T_j$.

For each component S of B, one of the four cases in the definition of degenerate maps happens. But case (1) cannot happen, for otherwise $f(S) \subset \varnothing$ after a homotopy. In case (3), $f_*(\pi_1(S))$ must be Z because $\pi_1(N)$ is torsion free. In case (4), using the relation $c^m = h$ between an exceptional fiber c and the regular fiber h, and the fact that $\pi_1(N)$ is torsion free, we conclude that $f_*(\gamma) = 1$ for all fibers γ. Let V be the base 2-manifold of the Seifert fibered manifold S, then $f_* : \pi_1(S) \to \pi_1(N)$ factors through $\pi_1(S)/<$ all fibers $> \cong \pi_1(V)$.

Define a group

$$\pi_S = \begin{cases} \{1\} & \text{case } (2), \\ Z & \text{case } (3), \\ \pi_1(V) & \text{case } (4), \end{cases}$$

and a space $D_S = K(\pi_S, 1)$. For the special case $V = P^2$, because $\pi_1(N)$ is torsion free, $f_*(\pi_1(S)) = 1$, so we put in case (2). Hence $H_3(\pi_1(V)) = 0$ for any V in our case (4). Therefore $H_3(D_S) = 0$ in all cases.

By the construction, there exist maps α_*, β_* on π_1 such that

$$\begin{array}{ccc} \pi_1(S) & \xrightarrow{(f|_S)_*} & \pi_1(N) \\ & {\scriptstyle\alpha_*}\searrow \quad \swarrow{\scriptstyle\beta_*} & \\ & \pi_1(D_s) & \end{array}$$

commutes. Since D_s and N are both $K(\pi, 1)$, the maps on π_1 are induced by maps $\alpha : S \to D_S$ and $\beta : D_S \to N$, resp.

For a technical reason, we choose α such that $\alpha|\partial S$ is nice in the following way: For each T in ∂S, there exists a base $\{\lambda, \mu\}$ of $\pi_1(T)$ such that $\alpha_*(\lambda) = 1$ in $\pi_1(D_S)$. We parametrize T by $T = S^1 \times S^1$, with $[S^1 \times *] = \lambda$, $[* \times S^1] = \mu$. We choose α such that $\alpha(x, y) = \alpha_1(y)$ for some embedding $\alpha_1 : S^1 \to D_S$ under this parametrization. Denote the knot $\alpha_1(S^1)$ by l_T. We may also assume that $l_{T_1} \cap l_{T_2} = \varnothing$ for different components T_1 and T_2 of ∂S by choosing D_S to be three dimensional.

Since $(\beta \circ \alpha)_* = (f|_S)_* : \pi_1(S) \to \pi_1(N)$, and N is aspherical, $f|_S$ is homotopic to $\beta \circ \alpha$. Extend this homotopy over M and replace f by the new map. Do this for each component S of B. And let $D_B = \bigcup D_S$, where the union is taken over all components S of B. Thus we have the commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{\ f\ } & N \\ & {\scriptstyle\alpha}\searrow \qquad \swarrow{\scriptstyle\beta} & \\ & D_B & \end{array}$$

Next we want to define α, β for G.

For each T on $\partial G = \partial S$, let λ be the simple closed curve defined as above. Attach a solid torus V to G along T such the meridian of V is identified with λ. Let $\widehat{G}$ denote the resulting closed manifold. Let the simple closed curve l'_T be the core of V which has the same orientation as μ.

Define $X = D_B \cup_\tau \widehat{G}$, where τ identifies each l_T in D_B with l'_T in $\widehat{G}$ preserving orientation. Define $\alpha : G \to \widehat{G}$ to be the obvious decomposition map which sends each $T \subset \partial G$ to l'_T, and maps the other part homeomorphically onto $\widehat{G} - l'_T$'s. It is easy to check that $\alpha : M = B \cup G \to X$ is a well-defined continuous map. Since $\alpha|_{G-\partial G} : G - \partial G \to \widehat{G} - \bigcup l'_T$ is a homeomorphism, we can define $\beta|_{\widehat{G}-\bigcup l'_T}$ to be $f \circ \alpha^{-1}$. So we get a map $\beta : X \to N$. The diagram

$$\begin{array}{ccc} M = B \cup G & \xrightarrow{\ f\ } & N \\ & {\scriptstyle\alpha}\searrow \qquad \swarrow{\scriptstyle\beta} & \\ & X = D_B \cup \widehat{G} & \end{array}$$

commutes by the construction.

Next we show how to get the manifold M_1 from X. Since $\deg f$ is odd, $f_*(H_3(M))$ is a subgroup of $H_3(N)$ of odd index, and therefore so is $\beta_*(H_3(X))$. Using the Mayer-Vietoris sequence and $H_3(D_B) = 0$ one can show that the map $i_* \colon H_3(\widehat{G}) \to H_3(X)$ is an isomorphism. Therefore, $(\beta \circ i)_*(H_3(\widehat{G})) = \beta_*(H_3(N))$. It follows that for some connected component M_1 of $\widehat{G}$, $\deg\{M_1 \overset{\beta\circ i}{\to} N\}$ is odd.

We will see that M_1 is a manifold such that $s(M_1) < s(M)$ and $H_{M_1} \cong H_M \cong H_N$ in the following argument:

M_1, as a component of $\widehat{G}$, is either

(a) $\widehat{T \times I}$, which is a lens space (including S^3 and $S^2 \times S^1$), or

(b) $\widehat{M_2}$, where M_2 is a union of some hyperbolic pieces and some Seifert fibered pieces connected together by some $T \times I$'s in $W_M \times I$.

In case (a), we have a map of nonzero degree from a lens space to N. The induced map on π_1 gives us a cyclic subgroup of $\pi_1(N)$ of finite index. But $\pi_1(N)$ contains $\pi_1(F)$, where F is some closed incompressible surface of genus > 0. This is a contradiction.

In case (b), for each $\widehat{S}$ in M_1, $f|_S$ is nondegenerate. Therefore for each ∂-component T of S, the λ defined as before is not a fiber of S by the definition of nondegenerate map. Hence the Seifert fibration of S extends to a Seifert fibration on $\widehat{S}$. Since $\pi_1(\widehat{S}) \cong \pi_S/\langle\lambda\rangle$ maps onto $\pi_1(S)/\ker f_* \cong f_*(\pi_1(S))$, and $f_*(\pi_1(S))$ is not cyclic by the definition of nondegenerate maps, $\widehat{S}$ is not a solid torus. So $\partial\widehat{S}$ (if it is not empty) is incompressible. If T is some torus which connects some S_1 and S_2 in M_2, then T also connects $\widehat{S}_1$ and $\widehat{S}_2$ in

M_1, and the fibers of $\widehat{S}_1$ and $\widehat{S}_2$ do not match up along T because the fibers of S_1 and S_2 do not match up along T. Therefore the torus decomposition of M_2 naturally gives the torus decomposition of M_1. So we have $s(M_1) = s(M_2)$. By the assumption in Case 2A, f is not degenerate on any $T\times I$ component of Σ_M, so f must be degenerate on some component S of S_M. Hence $s(M_2) < s(M)$. It follows that $s(M_1) = s(M_2) < s(M)$.

By the construction, $H_{M_1} \cong H_{M_2}$, which consists of some components of H_M. Using Gromov's norm and the nonzero degree map from M_1 to N, we see that H_{M_2} has to consist of all components of H_M. Hence $H_{M_1} \cong H_{M_2} \cong H_M \cong H_N$.

If $\beta|_{M_1} : M_1 \to N$ is a nondegenerate map, by the result of case (1) we have $s(M_1) \geq s(M_2)$. If $\beta|M_1$ is degenerate, $s(M_1) > s(N)$ by the induction hypothesis. Therefore, we have proved in any case that $s(M) > s(M_1) \geq s(N)$.

Case 2B. For some component S of Σ_M adjacent to some component H of H_M, $f|_S$ is degenerate.

Let T be a component of W_M, such that $T \times \{-1\}$ is a component of ∂S and $T \times \{1\}$ is a component of ∂H. In any one of the four cases in the definition of degenerate maps, there is a primitive element λ of $\pi_1(T)$ such that $f_*(\lambda) = 1$. Let $B = T\times[-1, 1]$, $G = \overline{M - B}$. Then $M = B\cup G$. Attach 2 solid tori V_1 and V_{-1} to G along $T\times\{1\}$ and $T\times\{-1\}$ respectively, so that meridians are identified with the two parallel copies of λ. Denote the resulting closed manifold by $\widehat{G}$. Using the same argument as in case 2A, we have a map β such that $\deg\{\widehat{G} \xrightarrow{\beta} N\} \neq 0$. So

$$\|N\| \leq \frac{1}{\deg \beta}\|\widehat{G}\| \leq \|\widehat{G}\|.$$

On the other hand, since the tori $(W_M - T) \cup T \times \{\pm 1\}$ cut $\widehat{G}$ into $V_{-1} \cup S_M \cup (H_M - H) \cup \widehat{H}$, by Theorem 1.8 we have

$$\begin{aligned}\|\widehat{G}\| &\leq \|V_{-1}\| + \|S_M\| + \|H_M - H\| + \|\widehat{H}\| = \|H_M\| - \|H\| + \|\widehat{H}\| \\ &< \|H_M\| - \|H\| + \|H\| = \|H_M\| = \|M\|.\end{aligned}$$

This implies that $\|N\| \leq \|\widehat{G}\| < \|M\|$ and therefore contradicts the assumption that $H_M \cong H_N$.

Lemma 2.14. *Suppose that M, N are oriented Haken manifolds with $H_M \cong H_N$, $S_M \cong S_N$, and a map $f: M \to N$ satisfies that*

(1) $\deg f = 1$.

(2) *f maps H_M onto H_N homeomorphically.*

(3) *f maps $(S_M, \partial S_M)$ onto $(S_N, \partial S_N)$, and for each component $S \ncong K\tilde{\times}I$ of S_M, f maps S homeomorphically onto some component S' of S_N.*

Then $M \cong N$.

Proof. After a homotopy of f supported on $W_M \times (-1, 1)$, $f^{-1}(W_N \times \{\pm 1\})$ is incompressible in $W_M \times [-1, 1]$. Thus it is a union of tori parallel to components of W_M. Let $c(f)$ be the number of components of $f^{-1}(W_N \times \{\pm 1\})$. Choose f among all the maps with the given property such that $c(f)$ is minimum.

Claim. $f^{-1}(W_N \times \{\pm 1\}) = W_M \times \{\pm 1\}$.

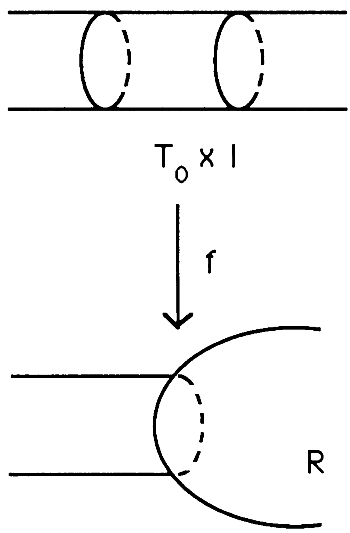

FIGURE 2

Proof of the Claim. Suppose that for some component T of $W_N \times \{\pm 1\}$, $f^{-1}(T)$ contains a component T_0 which is a parallel copy of a torus in $W_M \times (-1, 1)$. Let R be the component of $H_N \cup S_N$ such that $T \subseteq \partial R$. Since f is transverse to T, one side of T maps into R under f. Taking the connected component of $f^{-1}(R)$ containing this side, we get a "subinterval" $T_0 \times I \subseteq W_M \times (-1, 1)$ such that $(T_0 \times I, T_0 \times \partial I) \xrightarrow{f} (R, \partial R)$ (see Figure 2).

Case 1. $R \ncong K \tilde{\times} I$.

By Lemma 2.8, $f(T_0 \times \partial I) \subseteq T$, and we can modify $f|_{T_0} \times \mathring{I}$ by a homotopy such that $f(T_0 \times I) \subseteq T$. Push $f(T \times I)$ off R. Now $c(f)$ is decreased by 2. This is a contradiction.

Case 2. $R \cong K \tilde{\times} I$.

We consider the following two subcases.

Case 2A. $N \ncong K \tilde{\times} I \cup_\partial K \tilde{\times} I$.

Since $T_0 \times I \subseteq W_M \times (-1, 1)$, we may take T_1, T_2, T_3, T_4 to be the four consecutive tori in $f^{-1}(W_M \times \{\pm 1\})$ such that $[T_2, T_3] = T_0 \times I$ (see Figure 3).

Let T' be the parallel copy of T in $W_N \times \{-1, 1\}$. Then $T' \subseteq \partial R'$, where R' is a component of $H_N \cup S_N$. $R' \ncong K \tilde{\times} I$ because $N \ncong K \tilde{\times} I \cup_\partial K \tilde{\times} I$. By the result of Case 1, $f^{-1}(T')$ consists of only one component. So either T_1 or T_4, say T_1, does not map into T'. It follows that $f[T_1, T_2] \subseteq [T', T]$ but missing T'. Hence we can push $f([T_1, T_2])$ into R' to decrease $c(f)$ by 2.

Case 2B. $N \cong K \tilde{\times} I \cup_\partial K \tilde{\times} I$.

By assumption, $M \cong K \tilde{\times} I \cup_\partial K \tilde{\times} I$. Write $N = R_1 \cup_\partial R_2$, where $T = \partial R_1 = \partial R_2$. $f^{-1}(T)$ is a collection of parallel copies of tori in $W_M \times [-1, 1]$. Denote them by $T \times \{a_i\}$, $i = 1, \dots, n$. $\bigcup_i T \times \{a_i\}$ cuts M into two copies of $K \tilde{\times} I$, say Q_1, Q_2, and several copies of $T \times I$'s (see Figure 4).

Each "interval" $T \times [a_i, a_{i+1}]$ maps into either R_1 or R_2. By Lemma 2.8, either we can modify $f|T \times (a_i, a_{i+1})$ such that $f(T \times (a_i, a_{i+1})) \subset T = \partial R_1 = \partial R_2$, or we can homotope f such that $f|T \times [a_i, a_{i+1}]$ is a covering map onto

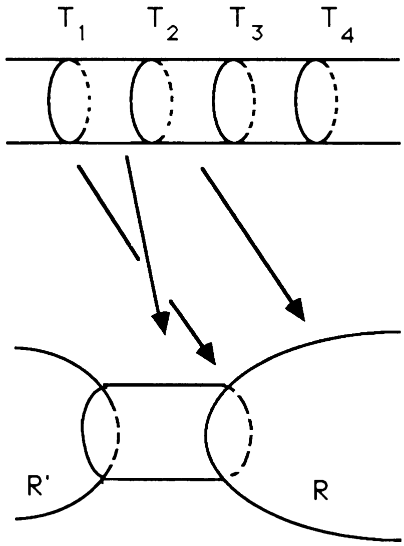

FIGURE 3

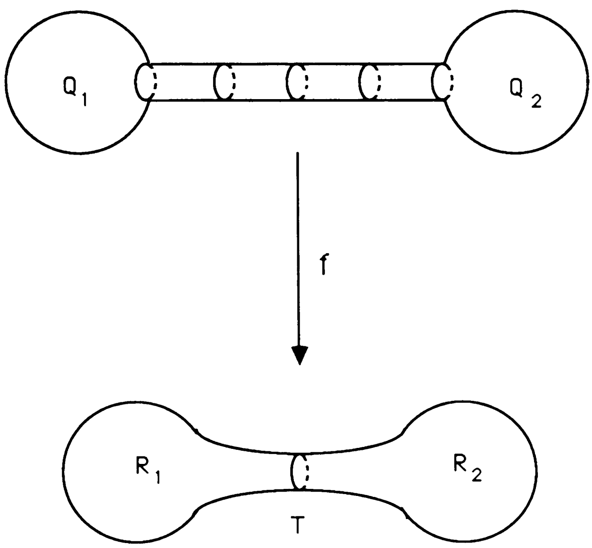

FIGURE 4

R_1 or R_2. The first case cannot happen because we can push $f(T \times [a_i, a_{i+1}])$ off R_i to decrease $c(f)$ by 2. Hence we can assume that $f|T \times [a_i, a_{i+1}]$ are all covering maps. It follows that

$$\deg\{T \times \{a_i\} \xrightarrow{f} T\} = \deg\{T \times \{a_{i+1}\} \xrightarrow{f} T\} = k$$

for all i. Therefore $\deg f = \deg\{f^{-1}(T) \to T\} = nk$. Since $\deg f = 1$, $n = k = 1$. So $f^{-1}(T)$ consists of one torus in W_M. By the transversality of f, we can take a small product neighborhood of T (resp. $f^{-1}(T)$) to be $W_N \times [-1, 1]$ (resp. $W_M \times [-1, 1]$) such that $f^{-1}(W_N \times \{\pm 1\}) = W_M \times (\{\pm 1\})$.

This finishes the proof of the claim.

A connectedness argument implies that each $T \times [-1, 1]$ in $W_M \times [-1, 1]$ is mapped into some $T \times [-1, 1]$ in $W_N \times [-1, 1]$. Hence for each $R \cong K\tilde{\times}I$ in S_N, $f^{-1}(R)$ consists of only one component Q in S_M, and $Q \cong K\tilde{\times}I$. Therefore $\deg\{Q \xrightarrow{f|_Q} R\} = 1$. By the Hopficity of $\pi_1(K\tilde{\times}I)$, $f|Q$ is homotopic to a homeomorphism. Extend this homotopy to a homotopy of f supported on a regular neighborhood of Q.

We then look at each component $T \times [-1, 1]$ of $W_M \times [-1, 1]$. $f(T \times [-1, 1]) = T_1 \times [-1, 1]$, some component of $W_N \times I$. Also f maps $T \times \{\pm 1\}$ homeomorphically into $T_1 \times \{\pm 1\}$. Since two homotopic homeomorphisms from a torus to a torus are isotopic, we can modify $f|T \times (-1, 1)$ such that f maps $T \times [-1, 1]$ onto $T_1 \times [-1, 1]$ homeomorphically.

Now our map f gives a bijection between the components of H_M, S_M, $W_M \times [-1, 1]$ and those of H_N, S_N, $W_N \times [-1, 1]$, and f is a homeomorphism on each of these components. This implies that f is a homeomorphism. Therefore $M \cong N$.

We are now ready to prove the theorem for the closed Haken manifold case.

Theorem 2.15. *Let $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots$ be an infinite sequence of maps between compact oriented 3-manifolds, such that*

(1) *each M_i is a closed Haken manifold.*

(2) $\deg f_i = 1$ *for each i.*

Then for i sufficiently large, $M_i \cong M_{i+1}$ and f_i is homotopic to a homeomorphism.

Proof. Since $\deg f_i = 1$, $\|M_1\| \geq \|M_2\| \geq \cdots$ by Theorem 1.5. By [19, 6.6.3], $\{\|M\| : M$ is a closed Haken manifold$\}$ is a well-ordered subset of the reals. So for i sufficiently large, $\|M_i\| = \|M_{i+1}\|$, and thus $v(H_{M_i}) = v(H_{M_{i+1}})$. By [19, 6.6.2], there are only finitely many hyperbolic manifolds (maybe disconnected) with a given volume. Therefore, by the remark after Theorem 2.1, we may pass to a subsequence such that $H_{M_1} \cong H_{M_2} \cong \cdots$. By Lemma 2.13, $s(M_1) \geq s(M_2) \geq \cdots$. So we can pass to another subsequence such that $s(M_1) = s(M_2) = \cdots$.

By Lemma 2.13, $(\Sigma_{M_i}, \varnothing) \xrightarrow{f_i} (M_{i+1}, \varnothing)$ is nondegenerate for all i. By Theorem 1.4, $f_i|_{\Sigma_{M_i}}$ is homotopic to a map f'_i such that $f'_i(\Sigma_{M_i}) \subseteq \Sigma_{M_{i+1}}$. Extend this to a homotopy over M_i so that after the homotopy $f_i(\Sigma_{M_i}) \subseteq \Sigma_{M_{i+1}}$. By Lemma 2.11, we can change f_i by a homotopy (fixing $f_i|\Sigma_{M_{i+1}}$) such that H_{M_i} maps onto $H_{M_{i+1}}$ homeomorphically under f_i.

For each component S of $S_{M_{i+1}}$, we can perturb f_i slightly such that f_i is transverse to ∂S and we may further assume that $f_i^{-1}(\partial S)$ is incompressible missing $H_M \cup \operatorname{Int}\Sigma_M$. So $f_i^{-1}(\partial S) = \bigcup T_j$, where each T_j is a parallel copy of torus in $W_M \times [-1, 1]$. $f_i^{-1}(S)$ is a union of some components of M_i cut along $f_i^{-1}(\partial S)$. Each such component is either a union of some components of S_M or some $T \times I$ in $W_M \times (-1, 1)$. If $S \ncong T \times I$, by Corollary 2.9, any map from $T \times I$ to S has degree either 0 or $2k$. Therefore $f_i^{-1}(S)$ contains at least one component of S_{M_i}. If $S \cong T \times I$, M_{i+1} is a fibered manifold with fiber a torus and a hyperbolic monodromy map. By passing to a subsequence, we may assume that all the manifolds in the sequence are such kind of fibered spaces. And the theorem in this case follows easily by showing that $f_i^{-1}(T)$

consists of one copy of fiber. Hence we assume now that the sequence does not contain such fibered spaces.

Since $s(M_i) = s(M_{i+1})$, $f_i^{-1}(S)$ contains exactly one component of S_{M_i}. So we can label the components of S_{M_i} as follows:

$$S_{M_i} = S_i^1 \cup \cdots \cup S_i^s, \quad \text{where } S_i^j \xrightarrow{f_i} S_{i+1}^j.$$

For each j, the sequence $\{S_i^j\}_i$ contains either finitely many or infinitely many $K\tilde{\times}I$'s. After passing to a subsequence, we may assume that each sequence either is a constant sequence $K\tilde{\times}I$ or does not contain $K\tilde{\times}I$ at all. For each j such that $\{S_i^j\}_i$ does not contain $K\tilde{\times}I$, the restricted map

$$(S_1^j, \partial S_1^j) \xrightarrow{f_1} (S_2^j, \partial S_2^j) \xrightarrow{f_2} \cdots$$

is a sequence of degree one maps. By Theorem 2.3, $f_i|S_i^j$ is homotopic to a homeomorphism for i large enough. Extend this to a homotopy of f_i which is supported on a regular neighborhood of S_i^j. We do this for all such j's. Now $f_i : M_i \to M_{i+1}$ satisfies the condition of Lemma 2.14, hence $M_i \cong M_{i+1}$. By Lemma 1.1, f_i induces an epimorphism on π_1. Therefore f_i induces an isomorphism by the Hopficity of $\pi_1(M_i)$. It follows that f_i is homotopic to a homeomorphism by Waldhausen's theorem.

3. Case of connected sum manifolds

3.1. **Manifolds without boundary.** We consider manifolds in $\mathcal{G}_c$. For any manifold $M = M_1 \# \cdots \# M_n$ in $\mathcal{G}_c$, M is a union of an n-punctured 3-sphere B_M and M_i' $(= M_i - B_i)$ glued along the 2-spheres S_i.

We define $n(M)$ to be the number of prime factors in the prime decomposition of M, and $r(M)$ to be the rank of $\pi_1(M)$. We call the spheres S_i a collection of splitting spheres on M.

For two p.l. 2-sided disjoint closed surfaces F_1, F_2 in M, we say that F_1 is weakly parallel to F_2, denoted by $F_1 \|_w F_2$, if one component of M cut along $F_1 \cup F_2$ is homeomorphic to $F_1 \times I \# X$ for some manifold X via a homeomorphism taking $F_1 \cup F_2$ to $F_1 \times \partial I$. We define the *weakly Haken-number* $h_w(M)$ to be the maximal number of p.l. 2-sided disjoint, nonweakly parallel, incompressible surfaces of genus > 0. Since this is no greater than the standard Haken-number, it must be finite.

Let $\mathcal{F}$ be a collection of maximal disjoint, nonweakly parallel, incompressible surfaces in M. Let F be the union of the surfaces in $\mathcal{F}$. If F intersects $\bigcup_i S_i$, we can take a disk D on S_i such that $\partial D = D \cap F$. Let F_1 be the component of $\mathcal{F}$ that gives this intersection. Let E be the disk on F_1 that ∂D bounds. We replace E by D and push D off S_i to get a surface F_1'. The new collection $F_1', F_2, \ldots, F_n$ is also a collection of maximal disjoint, nonweakly parallel, incompressible surfaces in M but intersecting $\bigcup S_i$ less. We can repeat this process until $F \cap \bigcup_i S_i = \varnothing$. So we have proved

Lemma 3.1. *In the above definition of weakly Haken number, we can choose the collection of surfaces missing* $\bigcup S_i$.

It follows that $h_w(M) = \sum h(M_i)$, where the summation is summed over all aspherical connected sum summands of M, and $h(M_i)$ is the Haken number of M_i.

For two p.l. 2-sided 2-submanifolds V_1 and V_2 in M we say that $V_1 \geq V_2$ if V_2 is gotten from V_1 by a finite number of following operations:

(a) Deleting a compressing S^2 (i.e. the boundary of a 3-ball) component from V_1 .

(b) If V_1 has a compressing disk D with regular neighborhood $D \times [-1, 1]$, we may delete the annulus $\partial D \times [-1, 1]$ from V_1, and cap off the resulting two boundary components by the two disks $D \times \{\pm 1\}$.

It is easy to see that if $V_1 \geq V_2$, then $[V_1] = [V_2]$ in $H_2(M)$.

Next, we give the definition of an almost defined map. Let M, N be 3-manifolds and $B_f = \bigcup_i B_i^+ \cup B_i^-$ is a finite collection of disjoint 3-ball pairs in $\operatorname{Int} M$. A map $f : M - \operatorname{Int} B_f \to N$ is called an *almost defined map* from M to N if for each i, $f|\partial B_i^+ = f|\partial B_i^- \circ r_i$ for some orientation reversing homeomorphism r_i from ∂B_i^+ to ∂B_i^-.

If the manifolds M and N are both oriented, and $f(\partial M) \subset \partial N$, we attach a copy of $S^2 \times [-1, 1]$ to $\partial B_i^+ \cup \partial B_i^-$ for each i to get an orientable manifold $M(f)$. Since $f|\partial B_i^+ = f|\partial B_i^- \circ r_i$, we can extend f to a proper map $\hat{f}$ from $M(f)$ to N such that $\hat{f}(S^2 \times [-1, 1]) = f(\partial B_i^+)$. We define the *degree* of f to be $\deg \hat{f}$. More generally, let M be an oriented manifold with boundary and τ to be an equivalence relation on ∂M such that the decomposition space M/τ is an oriented manifold. If $f : M \to N$ is a map which factors through a proper map on M/τ, then we define the *degree* of f to be the degree of the corresponding proper map from M/τ to N.

If f is an almost defined map between two closed manifolds M and N with $\pi_2(N) = 0$, then we can extend $f|\partial B_i^+ \cup \partial B_i^-$ over $B_i^+ \cup B_i^-$ to get a map f' from M to N. Since $f|\partial B_i^+ = f|B_i^- \circ r_i$, we can choose the extension f' such that $f'|B_i^+ = f|B_i^- \circ r_i$ for some homeomorphism r_i. It is easy to see that $\deg f' = \deg f$. So we have

Lemma 3.2. *Let M, N be closed oriented 3-manifolds with $\pi_2(N) = 0$. Then any almost defined map from M to N extends to a map on M with the same degree.*

In the next lemma, $\widehat{X}$ denotes the 3-manifold obtained from the 3-manifold X by capping off each S^2 component of ∂X with a 3-ball.

Lemma 3.3. *Let M, N be compact oriented 3-manifolds whose boundaries are disjoint unions of spheres. Assume also that $\pi_2(N) = 0$. Let $f : M - B_f \to N$ be an almost defined map. Then there is a map $\hat{f} : \widehat{M} \to \widehat{N}$ with $\deg \hat{f} = \deg f$.*

Proof. We add 3-handles to M to get $\widehat{M}$, and add 3-handles to N to get $\widehat{N}$. We extend $f|\partial M$ to the 3-handles so that the 3-handles are mapped into the 3-handles. Since this does not effect the preimage of any point in $\operatorname{Int} N$, the degree of f is not changed. We then apply Lemma 3.2 to extend the map over B_f.

For two almost defined maps f and g, we say that f is *B-equivalent* to g, if there are maps $f = f_0, f_1, \dots, f_n = g$ such that either f_i is homotopic to f_{i+1} (rel-$(\partial B_{f_i} \cup \partial B_{f_{i+1}})$) or $f_i = f_{i+1}$ on $M - B$ for a union of balls B containing $B_{f_i} \cup B_{f_{i+1}}$. This is a generalized definition of C-equivalent maps defined in [2].

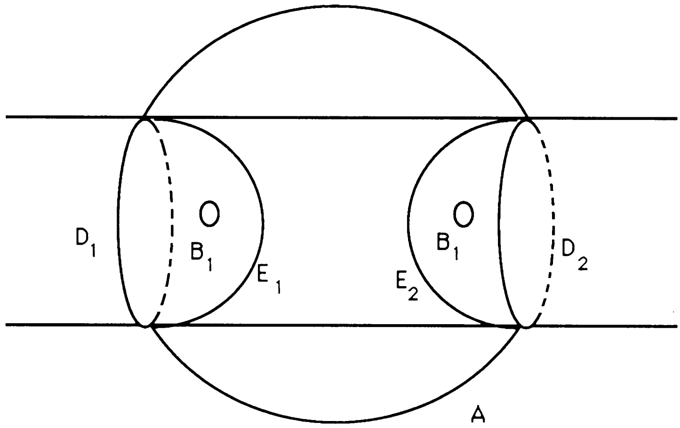

FIGURE 5

The next lemma is a modified version of Lemma 6.5 in [2] and Lemma III.9 in [4] by not requiring $\pi_2(F) = \pi_2(N - F) = 0$.

Lemma 3.4. *Let $f : M \to N$ be a map between two closed oriented 3-manifolds M and N. Let F be an incompressible 2-sided 2-submanifold in N. Then there is a collection of disjoint 3-balls B_g, and an almost defined map $g: M - B_g \to N$ such that*

(1) *g is B-equivalent to f.*
(2) *$g^{-1}(F)$ is a collection of incompressible surfaces in M.*
(3) $f^{-1}(F) \geq g^{-1}(F)$.
(4) $\deg g = \deg f$.

Proof. Homotope f such that f is transverse to F, thus $f^{-1}(F)$ consists of properly embedded 2-sided surfaces. Define the complexity of f to be

$$c(f) = (\dots, n_{-1}, n_0, n_1, n_2)$$

where n_i is the number of components of $f^{-1}(F)$ having Euler characteristic i. We order complexities lexicographically.

We prove that we can find g by an induction on $c(f)$.

If there is a compressing 2-sphere S^2 in $f^{-1}(F)$. Let B be the 3-cell bounded by S^2. If $F \cong S^2$, $[F] \neq 0$ in $\pi_2(N)$ since F is an incompressible S^2. The map $\pi_2(S^2) \xrightarrow{f|} \pi_2(F)$ must be a zero map because $f_*([S^2]) = f_*(0) = 0$ in $\pi_2(N)$. If $F \ncong S^2$, $\pi_2(F) = 0$, so the map on π_2 is still the zero map. In any case, we can redefine $f|B$ to get a new map f_1 such that $f_1(B) \subset F$. We then push $f_1(B)$ off F to eliminate the S^2 component from $f^{-1}(F)$.

If there is a 2-cell D in $\operatorname{Int} M$ with $D \cap f^{-1}(F) = \partial D$ and ∂D not contractible in $f^{-1}(F)$ (see Figure 5), we choose a regular neighborhood C of D in M such that $A = C \cap f^{-1}(F)$ is an annulus properly embedded in C. Let D_1 and D_2 be the disjoint 2-cells in ∂C with $\partial A = \partial D_1 \cup \partial D_2$, and choose disjoint 2-cells E_1 and E_2 properly embedded in C with $\partial E_i = \partial D_i$. $C - E_1 \cup E_2$ is a union of three 3-balls. From the interior of the left ball and the right ball, we choose a pair of small balls and denote the union of the two small balls by B_1 (Figure 5).

Define $f_1 : M - B_1 \to N$ as follows. On $M - \operatorname{Int} C$, define $f_1 = f$. Since $\ker(\pi_1(F) \to \pi_1(N)) = \{1\}$, we may extend $f_1|\partial E_i$ to map E_i into F. We may choose the map so that f maps the 2-sphere $E_1 \cup A \cup E_2$ trivially, and, $f(D_1 \cup E_1)$ and $f(D_2 \cup E_2)$ represent opposite elements in $[S^2, N]$. This can be done by defining $f|E_1$ to be $f|A \cup E_2$ composed with an orientation reversing map, and then push $f(E_1)$ into F. We now extend f to map the middle ball into $N - F$. Using the product structure on a neighborhood of F, we may extend f_1 to a map from $M - B_1$ to N such that f_1 satisfies the definition of an almost defined map and $f^{-1}(F) = (f^{-1}(F) - A) \cup E_1 \cup E_2$.

Note that in the above operation, we can choose the new map f_1 so that $f_1(C)$ is in a small neighborhood of $f(D) \cup F$. Therefore the preimage of any point "far away from" $f(D) \cup F$ is unchanged, and we conclude that $\deg f_1 = \deg f$.

One can check that $c(f_1) < c(f)$ and hence the lemma can be proved by an induction on $c(f)$.

The next lemma, which we will use in the proof of Lemma 3.6, is a slight generalization of Exercise I.35 of [4]. It can be proved using the generalized loop theorem. The proof is omitted.

Lemma 3.5. *Let M be a 3-manifold, and F_1, F_2 be two incompressible components of ∂M with F_1 closed. If for each loop l in F_1, $[l]^n$ is homotopic to a loop in F_2 for some $n > 0$, then $M \cong F_1 \times I \# X$ for some manifold X.*

Lemma 3.6. *Let $f : M \to N$ be a degree one map which is transverse to the splitting spheres $\bigcup_i S_i$ of N, then*

(1) *$h_w(M) \geq h_w(N)$.*

(2) *If $h_w(M) = h_w(N)$, then there is a map $g : M - B_g \to N$, B-equivalent to f, such that $\deg g = \deg f$ and $g^{-1}(\bigcup S_i)$ is a collection of 2-spheres.*

Proof. (1) Suppose that $h_w(M) < h_w(N)$. Let $\mathcal{F} = \{F_1, \ldots, F_k\}$ be a maximal collection of disjoint, nonweakly parallel, incompressible surfaces in N $(k = h_w(N))$. By Lemma 3.4, there is a map $g : M - B_g \to N$ such that $g^{-1}(F_i)$ is incompressible, and $\deg\{g^{-1}(F_i) \xrightarrow{g|} F_i\} = \deg f = 1$ for all i. Thus we can find a component V_i of $g^{-1}(F_i)$ such that $\deg\{V_i \xrightarrow{g|} F_i\} \neq 0$. By the assumption that $h_w(M) < h_w(N)$, two of the V_i's are weakly parallel in M, say $V_1 \|_w V_2$.

Let $V_1 \times I \# X$ be bounded by V_1 and V_2. We delete the X part to get $V_1 \times I$ minus a ball. Then we delete all the balls in B_g to get a punctured $V_1 \times I$. Denote it by $V_1 \times I - B$, where B is a disjoint union of balls in the interior of $V_1 \times I$. The incompressible surfaces $g^{-1}(F_1 \cup F_2)$ can be isotoped to miss ∂B. Hence each of its components either lies in or outside of $V \times I - B$.

Consider $g^{-1}(F_1 \cup F_2) \cap (V_1 \times I - B)$. If there is any incompressible S^2 in the intersection, the S^2 must bound a 3-ball in $V_1 \times I$, and we just delete a neighborhood of the ball to get a new $V_1 \times I - B$ to eliminate the S^2 component. If there is any incompressible surface V' of genus > 0 in $V_1 \times I - B$, Lemma 2.10 easily implies that V' is isotopic to $V_1 \times \{x\}$ for some $x \in I$. Say $g(V') \subset F_1$. Since $g : V_1 \to F_1$ and $g : V' \to F_1$ induces the same map on π_1, and since F_1 is aspherical, the two maps are homotopic, and thus have the same degree. By taking the rightmost component of such V', we get an "innermost interval" $V_1 \times [0, 1] - B$ such that g maps the ends $V_1 \times \{0, 1\}$ onto F_1, F_2 with nonzero degree, respectively, and $g(V_1 \times (0, 1) - B)$ does not intersect $F_1 \cup F_2$.

Let P be the component of N cut along $F_1 \cup F_2$ which $V\times[0,1]-B$ maps into under g. Since $\deg\{V_1\times\{0\} \xrightarrow{g|} F_1\} \neq 0$, the index $[\pi_1(F_1) : g_*\pi_1(V_1\times\{0\})]$ is finite by Lemma 1.3. Thus for any loop l in F_1, $[l]^n = g_*([a])$ for some $n > 0$ and some loop a in $V_1 \times \{0\}$. Since a is freely homotopic to a loop b in $V_1 \times \{1\}$, $[l]^n$ is freely homotopic to the loop $g_*(b)$ in F_2. By Lemma 3.5, $P \cong F_1 \times I\#X$ for some X. So we have proved that $F_1\|_w F_2$ in N, contradicting the assumption that the F_i's are not weakly parallel to each other.

Proof of (2). Similarly to the proof of (1), we have a map g, B-equivalent to f such that $g^{-1}((\bigcup S_i)\cup(\bigcup F_j))$ is incompressible in M, and $f^{-1}(S_i) \geq g^{-1}(S_i)$. By the proof of (1), $g^{-1}(\bigcup F_j)$ contains a maximal collection $\mathscr{V}$ of disjoint incompressible surfaces ($\not\cong S^2$) in M, and no component of $g^{-1}(\bigcup S_i)$ is weakly parallel to any component of $\mathscr{V}$. It follows that $g^{-1}(\bigcup S_i)$ is a collection of S^2's in M.

The following lemma is an easy corollary of Lemma 1.3.

Lemma 3.7. *Let N be a closed orientable 3-manifold and $f : S^3 \to N$ be a map. Then $\deg f = 0$ if $\pi_1(N)$ is infinite, and $\deg f \equiv 0 \pmod m$ if $\pi_1(N)$ is a finite group of order m.*

Next we consider a map $f : M \to N$ of degree one, where $M, N \in \mathscr{G}$. We also assume that $f : H_2(M) \to H_2(N)$ is an isomorphism, and $n(M) = n(N)$, $h_w(M) = h_w(N)$.

Lemma 3.8. *Under the above assumption, we can permute the prime factors $\{M_k\}$ of M such that there exist maps $g_k : M_k \to N_k$ satisfying*

(a) *If $\pi_1(N_k)$ is infinite, then $\deg g_k = 1$.*

(b) *If $\pi_1(N_k)$ is finite of order m, then $\deg g_k \equiv 1 \pmod m$.*

Proof. Let $\bigcup S_i$ be a set of splitting spheres of N, $\bigcup F_j$ be a maximal family of disjoint non weakly parallel incompressible surfaces ($\not\cong S^2$) in N missing $\bigcup S_i$, and $\bigcup S'_k$ be the set of nonseparating spheres in N. By Lemma 3.1, we may choose them to be disjoint from each other. Let f be transverse to $\bigcup S_i \cup S'_k \cup F_j$. By Lemma 3.6, there exists a map $g : M - B_g \to N$, B-equivalent to f with $\deg g = \deg f$, and $g^{-1}(\bigcup S_i \cup S'_k)$ is a set of spheres. Let R_{il} be the components of $g^{-1}(N'_i)$.

Since f induces an isomorphism on H_2, and $[S_i] = 0$ in $H_2(N)$, each component of $f^{-1}(S_i)$ is null-cobordant in M. Hence $g^{-1}(S_i)$ consists of null cobordant, thus separating spheres. So $\widehat{R_{ik}}$ is a connected summand of M. Since S'_k is not zero in $H_2(N)$, and since f induces a surjection on H_2, one of the spheres in $g^{-1}(S'_k)$ must be nonzero in $H_2(M)$, and thus is a nonseparating sphere.

Since $\deg g = \deg f = 1$, $\deg\{g^{-1}(N'_i) \xrightarrow{g|} N'_i\} = 1$, for each N'_i which is not homeomorphic to $S^2 \times S^1$. By Lemma 3.3, there exist maps $g_i : \bigcup \widehat{R_{ik}} \to N_i = \widehat{N}'_i$, such that $\deg g_i = 1$. By Lemma 3.7, each collection $\{\widehat{R_{ik}}\}$ cannot be just S^3's. Hence at least one of the R_{ik} is a nontrivial connected summand of M, denote it by $M_{\sigma(i)}$. For each N'_i which is homeomorphic to $S^2 \times S^1$, $g^{-1}(S'_i)$ contains a nonseparating sphere, and thus $g^{-1}(N'_i)$ must contain this nonseparating sphere, and thus it contains a $S^2 \times S^1$ as a connected summand of M. We take this as $M_{\sigma(i)}$. Since $n(M) = n(N)$, $\sigma(1), \dots, \sigma(n)$ is a permutation of $1, \dots, n$. So we have constructed the maps g_i from the connected summands of M into connected summands of N. The equations on the degree of g_i clearly follow from Lemma 3.7.

Theorem 3.9. *Let $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots$ be an infinite sequence of maps between compact oriented 3-manifolds such that for all i:*
(1) $M_i \in \mathcal{G}_c$,
(2) $\deg f_i = 1$.
Then for i sufficiently large, $M_i \simeq M_{i+1}$, and f_i is a homotopy equivalence.

Proof. Since $f_{i_*} : \pi_1(M_i) \to \pi_1(M_{i+1})$ is onto, $r(M_i) \geq r(M_{i+1})$. Since $n(M_i) \leq r(M_i)$, $n(M_i)$ is uniformly bounded for all i. Hence we can pass to a subsequence such that $n(M_1) = n(M_2) = \cdots$.

Since $f_{i_*} : H_2(M_i) \to H_2(M_{i+1})$ is onto, and $H_2(M_i)$ are all finitely generated abelian groups, we may assume that $f_{i_*} : H_2(M_i) \to H_2(M_{i+1})$ are all isomorphisms by passing to a subsequence.

By Lemma 3.6, $h_w(M_i) \geq h_w(M_{i+1})$, so we may assume that $h_w(M_1) = h_w(M_2) = \cdots$ by passing to a subsequence.

By Lemma 3.8, we can permute the factors of M_i's, such that $M_i = M_i^1 \# \cdots \# M_i^n$, and there is a map $g_i^k : M_i^k \to M_{i+1}^k$.

$$\deg g_i^k \begin{cases} = 1 & \text{if } \pi_1(M_{i+1}^k) \text{ is infinite}, \\ \equiv 1 \pmod{m} & \text{if } \pi_1(M_{i+1}^k) \text{ is of finite order } m. \end{cases}$$

For each k we consider the infinite sequence $\{M_i^k\}_i$. If $\pi_1(M_i^k)$ are all infinite for all i, the maps g_i^k's are all of degree one. Thus we apply results of §2 to conclude that $M_i^k \cong M_{i+1}^k$ for i sufficiently large. If one of the groups in $\{\pi_1(M_i^k)\}$'s is finite, then all the groups after this group are finite. So $\deg g_i^k = 1 + km$ for some integer k, where m is the order of $\pi_1(M_i^k)$. Let $s = [\pi_1(M_{i+1}^k) : g_{i_*}^k(\pi_1 M_i^k)]$. By Lemma 1.3, s divides $\deg g_i^k = 1 + km$. On the other hand, s divides $|\pi_1(M_{i+1}^k)| = m$. Hence we conclude that $s = 1$. This implies that g_i^k is onto, and therefore $|\pi_1(M_i^k)| \geq |\pi_1(M_{i+1}^k)|$. So we can pass to a subsequence such that for all k's such that $\{\pi_1(M_i^k)\}_i$ are all finite, $|\pi_1(M_i^k)| = |\pi_1(M_{i+1}^k)|$ for all i. Since there are only finitely many Seifert fibered manifolds with the same finite order in π_1, we can pass to a subsequence such that $M_i^k \cong M_{i+1}^k \cong \cdots$.

So we have proved that after passing to a subsequence $M_i \cong M_{i+1} \cong \cdots$. Thus the theorem is true by the remark after Theorem 2.1.

3.2. **Manifolds with boundary.** We define an equivalence relation $\sim$ on $\mathcal{G}$ by $M \sim N$ iff the pair $[M, \partial M]$ is homotopy equivalent to the pair $[N, \partial N]$. Define a relation $\geq$ on $\mathcal{G}/\sim$ by $[M] \geq [N]$ iff there is a degree one map $f : (M, \partial M) \to (N, \partial N)$. Since a homotopy equivalence of two manifold pairs with the same dimension is a degree one map, this is a well-defined relation on $\mathcal{G}/\sim$.

For each M in $\mathcal{G}$, we can form its double DM by $DM = M \cup_\partial M$. Let $M = \#_i M_i$, and each M_i is prime. For each M_i with $\partial M_i \neq \varnothing$ (thus M_i is Haken or a 3-ball), M_i can be cut along disks into 3-balls B_j and ∂-irreducible

Haken manifolds H_l. Let $\mathscr{D}$ be a maximal collection of such disjoint disks. Let $\mathscr{S}$ be the collection of spheres which cuts M into punctured M_i's. We cut DM along the spheres in $\mathscr{S}$ and the double of the disks in $\mathscr{D}$. Each remaining piece is either a punctured M_i where M_i is closed, or double of some ∂-irreducible Haken manifold, or a punctured 3-ball. So we have proved the following lemma:

Lemma 3.10. *$DM \in \mathscr{G}_c$ for all $M \in \mathscr{G}$.*

For a map $f : (M, \partial M) \to (N, \partial N)$, we can also define the double of f to be the obvious map $Df : DM \to DN$. There is an obvious inclusion map $i : M \to DM$. There is also an obvious retraction $r : DM \to M$ by identifying the two copies of M in DM.

Lemma 3.11. *Let $f : (M, \partial M) \to (N, \partial N)$ be a map of pairs such that $Df : (DM, \partial M) \to (DN, \partial N)$ is a homotopy equivalence of pairs. Then $f : (M, \partial M) \to (N, \partial N)$ is a homotopy equivalence of pairs.*

Proof. Let $G : (DN, \partial N) \to (DM, \partial M)$ be the homotopy inverse of Df. Thus $G \circ Df \simeq \mathrm{id}$ as a map of the pair $(DM, \partial M)$. Similarly for $Df \circ G$. Let $H_t : (DM, \partial M) \to (DM, \partial M)$ be maps such that $H_0 = \mathrm{id}$ and $H_1 = G \circ Df$.

Define a map $g : (N, \partial N) \to (M, \partial M)$ by $g = r_M \circ G \circ i_N$, and maps $h_t : (M, \partial M) \to (N, \partial N)$ by $h_t = r_M \circ H_t \circ i_M$. It is easy to check that $h_0 = \mathrm{id}$, and $h_1 = g \circ f$. Thus $g \circ f \simeq \mathrm{id}$. Similarly we can show that $f \circ g \simeq \mathrm{id}$. Hence $f : (M, \partial M) \to (N, \partial N)$ is a homotopy equivalence of pairs with homotopy inverse g.

Lemma 3.12. *Let $f : (M, \partial M) \to (N, \partial N)$ be a map of pairs such that $f|\partial : \partial M \to \partial N$ and $Df : DM \to DN$ are both homotopy equivalence. Then $Df : (DM, \partial M) \to (DN, \partial N)$ is a homotopy equivalence of pairs.*

Proof. Consider the homotopy exact sequence

$$\begin{array}{ccccccccc}
\pi_q(\partial M_1) & \xrightarrow{i_*} & \pi_q(DM_1) & \xrightarrow{j_*} & \pi_q(DM_1, \partial M_1) & \xrightarrow{\partial_*} & \pi_{q-1}(\partial M_1) & \xrightarrow{i_*} & \pi_{q-1}(DM_1) \\
\downarrow f|_* & & \downarrow Df|_* & & \downarrow Df|_* & & \downarrow f|_* & & \downarrow Df|_* \\
\pi_q(\partial M_2) & \xrightarrow{i_*} & \pi_q(DM_2) & \xrightarrow{j_*} & \pi_q(DM_2, \partial M_2) & \xrightarrow{\partial_*} & \pi_{q-1}(\partial M_2) & \xrightarrow{i_*} & \pi_{q-1}(DM_2)
\end{array}$$

the two left vertical maps and the two right vertical maps are both isomorphisms. By the Five-Lemma, the middle map is also an isomorphism. Thus Df is a homotopy equivalence of pairs by the relative version of Whitehead's theorem.

Combining Lemma 3.11 and Lemma 3.12, we have the following

Corollary 3.13. *If $f : (M, \partial M) \to (N, \partial N)$ is a map such that $f| : \partial M \to \partial N$ and $Df : DM \to DN$ are both homotopy equivalences, then $f : (M, \partial M) \to (N, \partial N)$ is a homotopy equivalence of pairs.*

Theorem 3.14. *The relation $\geq$ is a partial order on $\mathscr{G}/\sim$.*

Proof. Let M_1 and M_2 both be in $\mathscr{G}$ such that $[M_1] \geq [M_2]$ and $[M_2] \geq [M_1]$. Let $f_1 : (M_1, \partial M_1) \to (M_2, \partial M_2)$ and $f_2 : (M_2, \partial M_2) \to (M_1, \partial M_1)$ be two degree one maps. The maps Df_1 and Df_2 are both of degree one. Since DM_1 and DM_2 are both in $\mathscr{G}_c$, Df_1 is a homotopy equivalence by Theorem 2.1. f_1

and f_2 also induce degree one maps $f_1| : \partial M_1 \to \partial M_2$ and $f_2| : \partial M_2 \to \partial M_1$. This easily implies that $f_1| : \partial M_1 \to \partial M_2$ is also a homotopy equivalence. By Corollary 3.13, f_1 is a homotopy equivalence of pairs. So $[M] = [N]$.

Theorem 3.15. *Let* $(M_1, \partial M_1) \xrightarrow{f_1} (M_2, \partial M_2) \xrightarrow{f_2} \cdots$ *be an infinite sequence of maps between compact oriented* 3*-manifolds such that for all* i,

(1) $M_i \in \mathscr{G}$, *and*

(2) $\deg f_i = 1$.

Then for i *sufficiently large,* $(M_i, \partial M_i) \simeq (M_{i+1}, \partial M_{i+1})$, *and* f_i *is a homotopy equivalence.*

Proof. The doubles of the maps form an infinite sequence of degree one maps $DM_1 \xrightarrow{Df_1} DM_2 \xrightarrow{Df_2} \cdots$ between the manifolds DM_i in $\mathscr{G}_c$. By Theorem 3.9, the DF_i's are homotopy equivalences for i sufficiently large. Also it is easy to show that $f_i| : \partial M_i \to \partial M_{i+1}$ are homotopy equivalences for i sufficiently large. Therefore the conclusion of the theorem follows from Corollary 3.13.

As a corollary for Haken manifolds, we state

Corollary 3.16. *If there is an infinite sequence of degree one maps between oriented Haken* 3*-manifolds* $(M_1, \partial M_1) \xrightarrow{f_1} (M_2, \partial M_2) \xrightarrow{f_2} \cdots$, *then for* i *sufficiently large,* $M_i \cong M_{i+1}$ *and* f_i *is homotopic to a homeomorphism.*

References

1. M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math., No. 56, 1982, pp. 5–100.
2. J. Hempel, 3-*manifolds*, Ann. of Math. Stud., no. 86, Princeton Univ. Press, Princeton, N.J., 1976.
3. ——, *Residual finiteness for* 3-*manifolds*, Combinatorial Group Theory and Topology, Ann. of Math. Stud., no. 111, 1987, pp. 379–396.
4. W. Jaco, *Lectures on three-manifold topology*, CBMS Regional Conf. Ser. Math., no. 43, Amer. Math. Soc., Providence, R.I., 1980.
5. W. Jaco and P. Shalen, *Seifert fibered spacs in* 3-*manifolds*, Mem. Amer. Math. Soc., No. 220, 1979.
6. K. Johannson, *Homotopy equivalences of* 3-*manifolds with boundaries*, Lecture Notes in Math., vol. 761, Springer-Verlag, 1979.
7. F. Laudenbach, *Toplologie de la dimension trois, homotopie et isotopie*, Astérisque **12** (1974), 1–152.
8. J. Luecke, *Finite covers of* 3-*manifolds containing essential tori*, Trans. Amer. Math. Soc. **310** (1988), 381–392.
9. D. McCullough, *Mappings of reducible* 3-*manifolds*, M.S.R.I. preprint, 1984.
10. J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley, 1984.
11. P. Orlik, *Seifert manifolds*, Lecture Notes in Math., vol. 291, Springer-Verlag, 1972.
12. S. Plotnick, *Vanishing of Whitehead groups for Seifert manifolds with infinite fundamental groups*, Comment. Math. Helv. **55** (1980), 654–667.
13. P. Scott, *The geometry of* 3-*manifolds*, Bull. London Math. Soc. **15** (1983), 401–487.
14. T. Soma, *The Gromov invariant of links*, Invent. Math. **64** (1981), 445–454.
15. ——, *On preimage knots in* S^3, Proc. Amer. Math. Soc. **100** (1987), 587–592.
16. G. A. Swarup, *On embedded spheres in* 3-*manifolds*, Math. Ann. **203** (1973), 89–102.
17. C. B. Thomas, *The oriented homotopy type of compact* 3-*manifolds*, Proc. Amer. Math. Soc. **19** (1969), 31–44.

18. W. P. Thurston, *Three dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Amer. Math. Soc. **6** (1982), 357–381.

19. ——, *The geometry and topology of* 3-*manifolds*, Lecture Notes, Princeton Univ., 1979.

20. V. G. Turaev, *Towards the topological classification of geometric* 3-*manifolds* (Topology and Geometry Rohlin Seminar), Lecture Notes in Math., vol. 1346, 1988, pp. 291–323.

21. G. W. Whitehead, *Elements of homotopy theory*, Graduate Texts in Math., no. 61, Springer-Verlag, 1978.

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